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Spectral analysis of q-oscillator with general bilinear interaction

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Abstract. Spectra of the most general Hermitian Hamiltonian which is bilinear in the creation and annihilation operators of a q-harmonic oscillator are investigated with the help of the factorization method. It is shown that there are two factorization schemes leading to discrete spectra of complicated forms. For q-oscillator models with continuous spectrum the Hamiltonian may have normalizable eigenstates of infinite multiplicity. Existence of the continuous spectrum in the interacting system is also discussed.

1. Introduction

The harmonic oscillator is a basic quantum mechanical model whose importance spreads up to the quantum field theory. Intensive investigations of q-deformations of many physical constructions have been pursued during the last decade. These include various attempts to replace Lie algebras, describing spacetime structures or hidden symmetries, by their q-analogues called quantum algebras (or quantum groups). Certain aspects of the q-harmonic oscillator systems have been studied in [1–15]. An early analysis of the corresponding basic algebra has been performed in [16, 17].

In particular, in [12, 13] it was shown that the free non-relativistic particle can be interpreted as a q-harmonic oscillator without discrete spectrum (the limit $q \to 1$ is degenerate in this case). The behaviour of q-oscillators interacting linearly with a classical current was considered in [9, 14]. Such systems have interesting physical features. The number of bound states built as superpositions of q-Fock states decreases with the growth of the absolute value of the current. Once the value of the current exceeds a critical one, no normalizable states of this type are left [9]. In the q-oscillator models with a continuous spectrum [5, 6] there exists a second critical value of the current at which new bound states start to appear [14]. The qualitative property of these bound states is that they have an infinite multiplicity and that their number grows with the current.

In this present work we investigate a q-oscillator interacting with external currents through the general Hermitian form bilinear in creation and annihilation operators. This system is described by the abstract Hamiltonian

$$H = A^+A \qquad (1)$$

where

$$A = \alpha a + \beta a^+ + \gamma \qquad A^+ = \alpha^* a^+ + \beta^* a + \gamma^* \qquad (2)$$

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and the operators a^+ and a obey the q-harmonic oscillator algebra

$$aa^+ - qa^+a = 1 \qquad 0 < q < \infty. \tag{3}$$

The parameters α, β, γ are arbitrary complex numbers. We assume that a^+ is a Hermitian conjugate of a (this cannot be so if q is a complex number).

We define the 'free' q-oscillator Hamiltonian as

$$L = a^+a - \frac{1}{1-q}. \tag{4}$$

It satisfies the intertwining relations

$$La^+ = qa^+L \qquad La = q^{-1}aL. \tag{5}$$

One can consider (3), (5) as defining relations of an operator algebra generated by three formal operators a, a^+, L with some domains of definition in the Hilbert space. However, in analogy with the standard Heisenberg–Weyl algebra, it is natural to choose (3) as the only basic relation and consider (4) as a supplementary definition.

The general interacting system (1), (2) displays a rather complicated behaviour encoded through the interplay of 4 parameters α, β, γ, and q. The explicit form of H is rather lengthy:

$$\begin{aligned} H = (|\alpha|^2 + q|\beta|^2)a^+a + (\gamma^*\alpha + \beta^*\gamma)a + (\gamma\alpha^* + \beta\gamma^*)a^+ + \beta^*\alpha a^2 \\ +\alpha^*\beta a^{+2} + |\beta|^2 + |\gamma|^2. \end{aligned} \tag{6}$$

The main goal of this work is to study the eigenvalue problem for this operator, $H|\psi\rangle = \lambda|\psi\rangle$, and to classify the possible types of its spectra. The basic tool in our analysis is the factorization method [18], which has already been applied successfully to the $\alpha = 0$ (or $\beta = 0$) system [9, 14]. This paper essentially generalizes the corresponding results in two principally new points. First, for $\alpha \neq 0, \beta \neq 0$ dependence of the discrete spectrum λ_n on n takes a very complicated form which was not discussed in the literature before. Second, the model provides a highly non-trivial example of a system with more than one factorization of the Hamiltonian which brings in a further intrication of the spectrum. In general the situation is so complicated that we were actually able to perform only a partial analysis of the possible types of the energy spectra.

Let us consider the simplest unitary representation of the algebra (3)—the q-analogue of the Fock space. Suppose there exists a normalizable state annihilated by the operator a (the vacuum):

$$a|0\rangle = 0 \qquad \||0\rangle\|^2 \equiv \langle 0|0\rangle = 1. \tag{7}$$

(There are models where such a vacuum does not exist [12]—this is one of the principle differences between the standard and q-deformed harmonic oscillators.) It is not difficult to see that application of the powers of the operator a^+ to $|0\rangle$ generates eigenstates of the Hamiltonian L with exponential discrete (point) spectrum:

$$|n\rangle = \frac{(a^+)^n}{\sqrt{[n]_q!}}|0\rangle \qquad \langle n|m\rangle = \delta_{nm} \tag{8}$$

$$[n]_q! = [n]_q \cdot [n-1]_q! \qquad [0]_q! = 1 \qquad [n]_q = \frac{1-q^n}{1-q}$$

$$a^+|n\rangle = \sqrt{[n+1]_q}\;|n+1\rangle \qquad a|n\rangle = \sqrt{[n]_q}|n-1\rangle$$

$$L|n\rangle = -\frac{q^n}{1-q}|n\rangle \qquad n = 0, 1, 2, \ldots. \tag{9}$$

Eigenvalues of L for the q-Fock states $|n\rangle$ are positive for $q > 1$ and negative for $0 < q < 1$.

For $0 < q < 1$ the q-oscillator algebra (3) admits unitary representations for which eigenvalues of L are positive

$$L|\lambda\rangle = \lambda|\lambda\rangle \qquad \lambda > 0. \tag{10}$$

In contrast to the q-Fock states (8), in this case it is a^+ that plays the role of lowering operator (instead of the operator a):

$$a^+|\lambda\rangle\sqrt{q\lambda + \frac{1}{1-q}}|q\lambda\rangle \qquad a|\lambda\rangle = \sqrt{\lambda + \frac{1}{1-q}}|q^{-1}\lambda\rangle. \tag{11}$$

In distinction from (9), energies of the sequence of states generated by the action of powers of a^+ or a upon given $|\lambda_0\rangle$, where $\lambda_0 > 0$ is a free parameter, form a geometric progression

$$\lambda_n = \lambda_0 q^n \qquad n = 0, \pm 1, \pm 2, \ldots \tag{12}$$

which is infinite in both directions, i.e. there is an accumulation point $\lambda = 0$. The discrete spectrum of L may formally have an arbitrary number of such geometric progressions. The latter are known to be related to the bilateral q-hypergeometric series through the specific coherent states representation [12, 13].

It is possible that the abstract states (10) form a continuous spectrum [6, 12]. In this case they should be normalized as follows:

$$\langle\lambda_1|\lambda_2\rangle = \lambda_1\delta(\lambda_1 - \lambda_2). \tag{13}$$

One can remove the unusual λ_1 factor in front of the delta-function in this relation by redefinition of the states $|\lambda\rangle$. In this case it is necessary to change the definitions (11) appropriately. Here we assume that the states $|\lambda\rangle$ are not degenerate which is not always the case. The continuous spectrum may appear in the infinite-gap geometric series form [13], but for the free non-relativistic particle model which we use below, it occupies the whole half-axis $\lambda \geqslant 0$. Note that the states with $\lambda = 0$ form a special representation of the relations (3), (5), e.g. there are cases when a^+, a degenerate into c-numbers [6, 9].

In this paper, we analyse the eigenstates of H (6) formed by the q-Fock irreducible representation of the q-oscillator algebra (8). We also give a partial description of the spectra of H appearing from an infinite direct sum of the representations (11) determining the continuous spectrum of L. The cases when the states (11) belong to the point spectrum of L and the consequences for the spectra of H are not considered.

2. The factorization chain and formal discrete spectra

According to the standard factorization scheme [18], we should find a sequence of operators A_ℓ^+, A_ℓ and constants λ_ℓ which define a chain of Hamiltonians

$$H_\ell = A_\ell^+ A_\ell + \lambda_\ell \qquad \ell \in \mathbb{Z} \tag{14}$$

such that $H \equiv H_0$ and the neighbouring Hamiltonians, say H_ℓ and $H_{\ell+1}$, are connected to each other via the permutation of the operator factors in (14)

$$H_{\ell+1} = A_{\ell+1}^+ A_{\ell+1} + \lambda_{\ell+1} = A_\ell A_\ell^+ + \lambda_\ell. \tag{15}$$

This chain of operator equations, called the factorization chain, guarantees the relations

$$H_\ell A_\ell^+ = A_\ell^+ H_{\ell+1} \qquad A_\ell H_\ell = H_{\ell+1} A_\ell$$

which connect spectral properties of the Hamiltonians H_ℓ. Under certain conditions the numbers λ_ℓ generate the discrete spectrum of H_0. In these cases it is usually assumed that the zero modes of A_ℓ,

$$A_\ell|0\rangle_\ell^{\mathrm{int}} = 0 \tag{16}$$

provide the ground states of H_ℓ with the eigenvalues λ_ℓ

$$H_\ell|0\rangle_\ell^{\mathrm{int}} = \lambda_\ell|0\rangle_\ell^{\mathrm{int}}.$$

If $\lambda_\ell < \lambda_{\ell+1}$, then the ℓth eigenstate of H_0 is obtained by the action of a sequence of operators A_k^+ upon $|0\rangle_\ell^{\mathrm{int}}$:

$$\begin{aligned} |\ell\rangle_0^{\mathrm{int}} &= A_0^+ \dots A_{\ell-1}^+|0\rangle_\ell^{\mathrm{int}} \\ \||\ell\rangle_0^{\mathrm{int}}||^2 &= (\lambda_\ell - \lambda_0)\dots(\lambda_\ell - \lambda_{\ell-1})\||0\rangle_\ell^{\mathrm{int}}\|^2. \end{aligned} \tag{17}$$

From the latter relation it is seen that if $|0\rangle_\ell^{\mathrm{int}}$ is normalizable (i.e. if it belongs to the abstract Hilbert space) and

$$0 < (\lambda_\ell - \lambda_0)\dots(\lambda_\ell - \lambda_{\ell-1}) < \infty \tag{18}$$

then $|\ell\rangle_0^{\mathrm{int}}$ is normalizable as well. For some particular ℓ, the states $|\ell\rangle_0^{\mathrm{int}}$ can be normalizable even if the inequality $\lambda_k > \lambda_{k-1}$ is violated for some neighbouring λ_k's. In this case there are wave functions $|k\rangle_0^{\mathrm{int}}$ that do not belong to the Hilbert space and a more careful analysis is required to determine which of the λ_ℓ represent actual spectral points.

Since the factorization method is formulated in the abstract operator form, it is applicable to any eigenvalue problem. For the standard Schrödinger equation with analytical potentials, H_ℓ are differential operators of the second order. In this case there exist differential operators of the first order A_ℓ, such that the factorization chain (15) is satisfied. However, general existence statements are not constructive and our main interest lies in the opposite—in a search of some new solvable spectral problems via solutions of (15). Unfortunately, one often finds in this way only a part of the discrete spectrum.

Let us introduce the following sequence of operators A_ℓ^+, A_ℓ:

$$A_\ell = \alpha_\ell a + \beta_\ell a^+ + \gamma_\ell \qquad A_\ell^+ = \alpha_\ell^* a^+ + \beta_\ell^* a + \gamma_\ell^*. \tag{19}$$

Here the integer index ℓ takes values from zero up to some number N determined by the normalizability condition on the zero modes of the operators A_ℓ. All Hamiltonians H_ℓ have evidently the form (6).

Proposition 1. The operators (19) provide a solution of the factorization chain (15) for a particular choice of the parameters $\alpha_\ell, \beta_\ell, \gamma_\ell$. This gives the formal discrete spectrum λ_ℓ of the Hamiltonian (6)

$$\begin{aligned} \lambda_\ell = \frac{(\alpha_0 q^{\ell/2} + \beta_0 q^{-\ell/2})^2}{q-1} - \frac{(\alpha_0+\beta_0)^2(\mathrm{Re}\,\gamma_0)^2}{(\alpha_0 q^{\ell/2} + \beta_0 q^{-\ell/2})^2} \\ - \frac{(\alpha_0-\beta_0)^2(\mathrm{Im}\,\gamma_0)^2}{(\alpha_0 q^{\ell/2} - \beta_0 q^{-\ell/2})^2} + |\gamma_0|^2 + \lambda_0 - \frac{(\alpha_0+\beta_0)^2}{q-1} \end{aligned} \tag{20}$$

where $\ell = 0, 1, \dots, N$ and $\alpha_0, \beta_0, \gamma_0, \lambda_0$ are parameters related to α, β, γ and subject to the constraint $\beta_0/\alpha_0 \neq \pm q^k$ for some integer $k \geqslant 0$.

Proof. Substituting (19) into (15) and using (3) we find the following system of equations for α_ℓ, β_ℓ, γ_ℓ, and λ_ℓ:

$$\begin{aligned}
&|\alpha_{\ell+1}|^2 + q|\beta_{\ell+1}|^2 = q|\alpha_\ell|^2 + |\beta_\ell|^2 \\
&\alpha_\ell \beta_\ell^* = \alpha_{\ell+1}\beta_{\ell+1}^* \\
&\gamma_\ell \alpha_\ell^* + \gamma_\ell^* \beta_\ell = \gamma_{\ell+1}\alpha_{\ell+1}^* + \gamma_{\ell+1}^*\beta_{\ell+1} \\
&\lambda_\ell + |\alpha_\ell|^2 + |\gamma_\ell|^2 = \lambda_{\ell+1} + |\beta_{\ell+1}|^2 + |\gamma_{\ell+1}|^2.
\end{aligned} \tag{21}$$

The gauge transformations $a \to a\mathrm{e}^{\mathrm{i}\theta}$ and $A \to A\mathrm{e}^{\mathrm{i}\varphi}$ allow us to take α and β as real numbers in the Hamiltonian (1) without loss of generality. It then follows from the above equations that we may also take α_ℓ and β_ℓ a real. This choice leads to the following non-trivial solution of (21):

$$\begin{aligned}
&\alpha_\ell = \alpha_0 q^{\ell/2} \qquad \beta_\ell = \beta_0 q^{-\ell/2} \\
&\gamma_\ell = \frac{(\alpha_0+\beta_0)\mathrm{Re}\,\gamma_0}{\alpha_\ell+\beta_\ell} + \frac{\mathrm{i}(\alpha_0-\beta_0)\mathrm{Im}\,\gamma_0}{\alpha_\ell-\beta_\ell}
\end{aligned} \tag{22}$$

with the rather involved formula for the discrete spectrum (20). Actually, one can simultaneously change the signs of $\alpha_\ell, \beta_\ell, \gamma_\ell$ for each ℓ, but such a freedom is obvious and may be discarded. Note that if one has $\beta_0/\alpha_0 = \pm q^k$, k a positive integer or zero, then, unless $\mathrm{Re}\,\gamma_0 = 0$ or $\mathrm{Im}\,\gamma_0 = 0$, the eigenvalues (20) diverge for $\ell = k$. In these situations the factorization method cannot be applied directly through the ansatz (19).

There are two possible identifications of H_0 as given in (14) with H as specified in (1). The first one is obvious:

$$\alpha_0 = \alpha \qquad \beta_0 = \beta \qquad \gamma_0 = \gamma \qquad \lambda_0 = 0. \tag{23}$$

This corresponds to the following sequence of parameters

$$\begin{aligned}
&\alpha_\ell = \alpha q^{\ell/2} \qquad \beta_\ell = \beta q^{-\ell/2} \\
&\gamma_\ell = \frac{(\alpha+\beta)\mathrm{Re}\,\gamma}{\alpha q^{\ell/2}+\beta q^{-\ell/2}} + \frac{\mathrm{i}(\alpha-\beta)\mathrm{Im}\,\gamma}{\alpha q^{\ell/2}-\beta q^{-\ell/2}}.
\end{aligned}$$

Another identification corresponds to the choice $\lambda_0 \neq 0$:

$$\begin{aligned}
&\alpha_0 = \beta q^{1/2} \qquad \beta_0 = \alpha q^{-1/2} \\
&\gamma_0 = \frac{(\alpha+\beta)\mathrm{Re}\,\gamma}{\alpha q^{-1/2}+\beta q^{1/2}} - \frac{i(\alpha-\beta)\mathrm{Im}\,\gamma}{\alpha q^{-1/2}-\beta q^{1/2}} \\
&\lambda_0 = \left(\beta^2 - \frac{\alpha^2}{q}\right)\left(1 + \frac{(\mathrm{Re}\,\gamma)^2(q-1)}{(\alpha q^{-1/2}+\beta q^{1/2})^2} + \frac{(\mathrm{Im}\,\gamma)^2(q-1)}{(\alpha q^{-1/2}-\beta q^{1/2})^2}\right).
\end{aligned} \tag{24}$$

These initial conditions yield:

$$\begin{aligned}
\alpha_\ell &= \beta q^{(\ell+1)/2} \qquad \beta_\ell = \alpha q^{-(\ell+1)/2} \\
\gamma_\ell &= \frac{(\alpha+\beta)\mathrm{Re}\,\gamma}{\alpha q^{-(\ell+1)/2}+\beta q^{(\ell+1)/2}} + \frac{\mathrm{i}(\alpha-\beta)\mathrm{Im}\,\gamma}{\beta q^{(\ell+1)/2}-\alpha q^{-(\ell+1)/2}} \\
\lambda_\ell &= \frac{(\beta q^{(\ell+1)/2}+\alpha q^{-(\ell+1)/2})^2}{q-1} - \frac{(\alpha+\beta)^2(\mathrm{Re}\,\gamma)^2}{(\beta q^{(\ell+1)/2}+\alpha q^{-(\ell+1)/2})^2} \\
&\quad - \frac{(\alpha-\beta)^2(\mathrm{Im}\,\gamma)^2}{(\beta q^{(\ell+1)/2}-\alpha q^{-(\ell+1)/2})^2} + |\gamma|^2 + \frac{(\alpha+\beta)^2}{1-q}.
\end{aligned}$$

There are thus two types of factorization schemes, or two possible branches of the discrete spectrum λ_ℓ. $\square$

The structure of the Hamiltonian (6) is seen clearly in the following renormalized expression

$$\begin{aligned} L_{\mathrm{int}} &= \frac{1}{\alpha_0^2 + q\beta_0^2}\left(H_0 - \lambda_0 - |\gamma_0|^2 - \frac{\alpha_0^2+\beta_0^2}{1-q}\right) \\ &= a^+a + ja^+ + j^*a + J(a^{+2}+a^2) - \frac{1}{1-q} \end{aligned} \tag{25}$$

where

$$j = \frac{\gamma_0\alpha_0 + \beta_0\gamma_0^*}{\alpha_0^2 + q\beta_0^2} \qquad J = \frac{\alpha_0\beta_0}{\alpha_0^2 + q\beta_0^2}.$$

The energy scale was chosen in such way that at $j = J = 0$ one gets the 'free' system (4). For arbitrary values of α_0 and β_0 the current J varies in the range

$$0 \leqslant |J| \leqslant \frac{1}{2\sqrt{q}}. \tag{26}$$

For $|J| \geqslant \frac{1}{2}\sqrt{q}$ the Hamiltonian (25) cannot be factorized, which indicates an instability of the system in this region. Let us introduce the parametrization $q \equiv \mathrm{e}^{2\omega}$. The formal discrete spectrum of the operator (25) can then be written as follows:

$$E_\ell = \frac{2J}{q-1}\cosh 2\omega(\ell+\sigma) - \frac{1}{4J}\left(\frac{(\mathrm{Re}\, j)^2}{\cosh^2\omega(\ell+\sigma)} + \frac{(\mathrm{Im}\, j)^2}{\sinh^2\omega(\ell+\sigma)}\right) \tag{27}$$

for $J > 0$, and

$$E_\ell = \frac{2|J|}{q-1}\cosh 2\omega(\ell+\sigma) - \frac{1}{4|J|}\left(\frac{(\mathrm{Re}\, j)^2}{\sinh^2\omega(\ell+\sigma)} + \frac{(\mathrm{Im}\, j)^2}{\cosh^2\omega(\ell+\sigma)}\right) \tag{28}$$

for $J < 0$. The parameter σ,

$$\sigma = \frac{1}{2\omega}\ln\frac{|\alpha_0|}{|\beta_0|} \tag{29}$$

depends on the ratio α_0/β_0 and is not uniquely determined from a fixed J. (This is why there are two branches in the spectrum, or two factorization schemes). We parametrize the first scheme by the choice

$$\sigma = \sigma_1 = \frac{1}{2\omega}\ln\frac{1+\sqrt{1-4J^2q}}{2|J|} \tag{30}$$

and the second one by

$$\sigma = \sigma_2 = \frac{1}{2\omega}\ln\frac{1-\sqrt{1-4J^2q}}{2|J|} = 1 - \sigma_1. \tag{31}$$

Note that when J varies between 0 and $\frac{1}{2}\sqrt{q}$ and $q > 1$ the parameter σ_1 changes from ∞ to $\frac{1}{2}$ and σ_2 changes from $-\infty$ to $\frac{1}{2}$. In the latter case one passes the values $\sigma = -k, k = 0, 1, 2, \ldots$ at which the formal spectrum (27) (resp. (28)) explodes unless $\mathrm{Im}\, j = 0$ (resp. $\mathrm{Re}\, j = 0$) as was mentioned already. For $q < 1$ the ranges of σ_1 and σ_2 interchange.

We are now ready to analyse the spectral properties of our system at different values of the parameters.

3. Discrete spectrum for $q > 1$

Let us first analyse the situation when $q > 1$ (or $\omega > 0$). In this case all the operators coming into play are unbounded. Our system has an infinite unbounded discrete spectrum independently on the parameters α, β, and γ.

Indeed, consider the zero modes of the operators A_ℓ (16) which are formal eigenstates of the Hamiltonian (1) with eigenvalues λ_ℓ. The normalizability of these states can be determined with the help of the expansion

$$|0\rangle_\ell^{\text{int}} = \sum_{n=0}^{\infty} B_n(\ell)|n\rangle, \tag{32}$$

where $|n\rangle$ are the q-Fock states of the non-interacting q-oscillator Hamiltonian L obtained from (25) by setting $j = J = 0$. Substituting (32) into (16) and suppressing the dependence of B_n on ℓ, one arrives at the recursion relation

$$\alpha_\ell\sqrt{[n+1]_q}B_{n+1} + \beta_\ell\sqrt{[n]_q}B_{n-1} + \gamma_\ell B_n = 0 \qquad n = 1, 2, \ldots \tag{33}$$

with the initial condition $B_1 = -\gamma_\ell B_0/\alpha_\ell$. Assuming $\alpha_\ell\beta_\ell > 0$, one may write

$$B_n = \left(\frac{\beta_\ell}{\alpha_\ell}\right)^{n/2}\frac{P_n(z_\ell)}{\sqrt{(-1)^n(q;q)_n}} \qquad z_\ell = -\gamma_\ell\sqrt{\frac{q-1}{\alpha_\ell\beta_\ell}}$$

where the standard notation for q-shifted factorial [19] is used:

$$(a;q)_0 = 1 \qquad (a;q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1}).$$

Substituting this Ansatz into (33), one sees [20] that $P_n(z_\ell)$ coincide with the monic continuous q-Hermite polynomials for $q > 1$ satisfying the three-term recurrence relation

$$P_{n+1}(z) + (q^n - 1)P_{n-1}(z) = zP_n(z).$$

Note that in our case the argument of the polynomials z_ℓ is complex. For $\alpha_\ell\beta_\ell < 0$ one derives similar formulae after the transformation $P_n(z) \to \mathrm{i}^n P_n(z/\mathrm{i})$.

In order for the ground state $|0\rangle_\ell^{\text{int}}$ to be stable it is necessary for the condition

$$\||0\rangle_\ell^{\text{int}}\|^2 = \sum_{n=0}^{\infty}|B_n|^2 < \infty \tag{34}$$

to be satisfied. In other words it is necessary that $B_n \to 0$ for $n \to \infty$ sufficiently fast. Writing asymptotic form of (33) as $n \to \infty$

$$\alpha_\ell q^{1/2}B_{n+1} + \beta_\ell B_{n-1} = 0 \qquad q > 1$$

we see that

$$\frac{|B_{n+1}|}{|B_{n-1}|} \to \frac{|\beta_\ell|}{|\alpha_\ell|q^{1/2}}.$$

Therefore the condition of normalizability of the states $|0\rangle_\ell^{\text{int}}$ is equivalent to the following constraint:

$$|\beta_0/\alpha_0| < q^{\ell+1/2}.$$

According to (29) this means that one should have

$$\sigma(J) > -\ell - \tfrac{1}{2}. \tag{35}$$

Proposition 2. Eigenvalues E_ℓ determined by the first factorization scheme, i.e. appearing from (27), (28) for $\ell = 0, 1, \ldots, \infty$ after the substitution $\sigma = \sigma_1$, belong to the point spectrum of the Hamiltonian L_{int}, i.e. the corresponding eigenstates (32) lie in the abstract Hilbert space.

This statement follows from the fact that $\sigma_1(J) > 0$ for all the allowed values of J, which evidently satisfies the condition (35). Note that in the limit $J \to 0$ this scheme gives the discrete spectrum of the $J = 0$ model [9, 14]:

$$E_\ell = \frac{q^\ell}{q-1} - |j|^2 q^{-\ell} \qquad \ell = 0, 1, \ldots, \infty.$$

Consideration of the second factorization scheme is more complicated. For a fixed J one finds that the states $|0\rangle_\ell^{\text{int}}$ are normalizable for values of ℓ that start at $\ell = k$, where k is the minimal integer such that $k > -\frac{1}{2} - \sigma_2$. In order for λ_ℓ, $\ell \geqslant k$, to belong to the spectrum of H_0 the positivity condition (18) must be satisfied. For $\ell \geqslant 2k + 1$ it is always fulfilled. In the region $k \leqslant \ell < 2k + 1$ the positivity takes place only for odd or even ℓ depending on the value of σ_2. Thus the spectrum is complicated for $\sigma_2 < -\frac{1}{2}$: it contains all λ_ℓ for $\ell > 2k$ and a sieved part of λ_ℓ for $k \leqslant \ell \leqslant 2k$. Take for example, $k = 1$. Then, the state $|1\rangle_0^{\text{int}}$ is not normalizable since $\lambda_1 - \lambda_0 < 0$; but the state $|2\rangle_0^{\text{int}}$ belongs to the spectrum when $-1 < \sigma_2 < -\frac{1}{2}$ because then $(\lambda_2 - \lambda_0)(\lambda_2 - \lambda_1) > 0$. For $-\frac{3}{2} < \sigma_2 < -1$, the positivity condition is violated, i.e. $|2\rangle_0^{\text{int}}$ is not physical. A similar situation takes place for $k > 1$. Note, however, that in order to pass from one region where such a behaviour is observed to another it is necessary to cross the singular points $\sigma_2 = -k$ where one of the parameters λ_ℓ entering the factorization chain diverges. The latter singularities are present for $\operatorname{Im} j \neq 0$ $(J > 0)$ or $\operatorname{Re} j \neq 0$ $(J < 0)$ which indicates some instability. At the critical points $J = q^k/(1 + q^{2k+1})$ (when $\sigma_2 = -k$) and for generic complex current j, the discrete spectrum is determined only from the first factorization scheme.

Note that the lowest bound state energy λ_0 in (24) is negative, $\lambda_0 < 0$, when $q^{-3/2} < |\beta/\alpha| < q^{-1/2}$, which is in contradiction with the formal condition of positivity of the Hamiltonian (1). This contradiction is resolved by the observation that for $|\beta/\alpha| < q^{-1/2}$ the zero mode of A_0 in the second factorization scheme does not belong to the domain of definition of the unbounded operator A_0 of the first factorization scheme. If $|\beta/\alpha| > q^{-1/2}$, i.e. if $\lambda_0 > 0$, then the application of A_0 of the first factorization scheme to the ground state of the second scheme gives a normalizable state.

As an illustration, let us consider the coherent states representation of the q-harmonic oscillator algebra [17]

$$a = z^{-1}\frac{1-D}{1-q} \qquad a^+ = z \qquad Df(z) = f(qz). \tag{36}$$

It yields the following second order q-difference equation as the eigenvalue problem for the operator L_{int}:

$$\begin{aligned} &\frac{Jq^{-1}}{(1-q)^2}\psi(q^2 z) - \frac{1}{1-q}\left(z^2 + zj^* + \frac{J(1+q^{-1})}{1-q}\right)\psi(qz) \\ &\qquad + \left(jz^3 + \frac{j^* z}{1-q} + J\left(z^4 + \frac{1}{(1-q)^2}\right) - \lambda z^2\right)\psi(z) = 0. \end{aligned} \tag{37}$$

In this case investigation of the discrete spectrum eigenfunctions appears to be not very difficult.

Proposition 3. Orthonormal eigenfunctions $\psi^{(n)}(z)$ of the operator H (6) corresponding to the eigenvalues λ_n (20) have the explicit form

$$\psi^{(n)}(z) = C_n P_n(z) \prod_{k=1}^{\infty} \left(1 + \frac{(1-q)\gamma_n z}{\alpha_n q^k} + \frac{(1-q)\beta_n z^2}{\alpha_n q^{2k}}\right) \tag{38}$$

where $P_n(z)$ is a nth order polynomial of z

$$P_n(z) = \prod_{i=0}^{n-1} \left[\frac{\gamma_i^*}{\beta_i} - \frac{\gamma_n}{\alpha_n} + \left(\frac{\alpha_i}{\beta_i} - \frac{\beta_n}{\alpha_n}\right) z\right] \tag{39}$$

and C_n are the normalization constants

$$C_n = C_0 \prod_{i=0}^{n-1} \beta_i (\lambda_n - \lambda_i)^{-1/2} \qquad n > 0.$$

This statement follows from the fact that the infinite product standing on the right hand side of (38) determines the unique zero mode of the operator A_n. Using the relation (17) and performing simple calculations we arrive at the above formulae.

4. Discrete spectrum for $q < 1$

Now consider the situation when $0 < q < 1$ (or $\omega < 0$). We start by determining the normalizability conditions of the Hamiltonian eigenstates $|0\rangle_\ell^{\text{int}}$ (16) using again the expansion (32). Substituting the ansatz (with $\alpha_\ell \beta_\ell > 0$ understood)

$$B_n(\ell) = \left(\frac{\beta_\ell}{\alpha_\ell}\right)^{n/2} \frac{P_n(z_\ell)}{\sqrt{(q;q)_n}} \qquad z_\ell = -\gamma_\ell \sqrt{\frac{1-q}{\alpha_\ell \beta_\ell}}$$

into (33), one arrives at the three-term recurrence relation for the continuous q-Hermite polynomials [20]:

$$P_{n+1}(z_\ell) + (1-q^n) P_{n-1}(z_\ell) = z_\ell P_n(z_\ell).$$

For $n \to \infty$, equation (33) simplifies to

$$\alpha_\ell B_{n+1} + \beta_\ell B_{n-1} + \gamma_\ell \sqrt{1-q}\, B_n = 0. \tag{40}$$

Its general solution is

$$B_n = c_1 \zeta_1^n + c_2 \zeta_2^n \qquad \zeta_{1,2} = \frac{-\gamma_\ell \sqrt{1-q} \pm \sqrt{(1-q)\gamma_\ell^2 - 4\alpha_\ell \beta_\ell}}{2\alpha_\ell}.$$

Consequently, the state $|0\rangle_\ell^{\text{int}}$ shall be normalizable only if $|\zeta_1| < 1$ or $|\zeta_2| < 1$, and, additionally, if the solution of (33) satisfying the initial condition $B_1 = -\gamma_\ell B_0/\alpha_\ell$ is asymptotically proportional to ζ_1^n or ζ_2^n respectively as $n \to \infty$.

Analysis of the general situation is thus quite complicated. Let us discuss briefly spectra of the Hamiltonian (25) with

$$j = 0 \qquad \text{or} \qquad \gamma_\ell = 0.$$

From the asymptotic equation $\alpha_\ell B_{n+1} + \beta_\ell B_{n-1} = 0$ we see that the normalizability conditon is equivalent to the constraint

$$|\beta_\ell/\alpha_\ell| < 1 \qquad \text{or} \qquad |\beta_0/\alpha_0| < q^\ell. \tag{41}$$

This is equivalent to the requirement $\sigma < -\ell$. As a result we see that only the first factorization scheme generates the discrete spectrum provided the condition

$$|J| < \frac{q^{\ell}}{1+q^{2\ell+1}}$$

is satisfied. Since for sufficiently large ℓ this inequality is violated, the number of bound states is finite and equals to the integral part of $1-\sigma_1$.

As was mentioned, a simple form of our 'Schrödinger equation' is obtained in the coherent states representation (36). Repeating the considerations performed for $q > 1$ in the section 3, we arrive at the following representation of formal eigenstates of the Hamiltonian H with eigenvalues λ_n:

$$\psi^{(n)}(z) = C_n P_n(z) \prod_{k=0}^{\infty} \left(1 + \frac{(1-q)\gamma_n z}{\alpha_n} q^k + \frac{(1-q)\beta_n z^2}{\alpha_n} q^{2k}\right)^{-1}$$

where $P_n(z)$ is the polynomial fixed in (39) and C_n are the normalization constants.

It could be thought that the problem of qualitative characterization of the spectrum classes of our system is completed, but, in fact, only a part of it has been treated. New qualitative features appear in the situation when $q < 1$ and the 'free' q-oscillator has both infinite discrete and continuous spectra with positive energies [5, 6, 13].

The continuous spectrum may occupy some parts of the half-axis $0 < \lambda < \infty$. If it does not cover this region completely, then it appears in the form of infinitely many bands that accumulate near the $\lambda = 0$ point from above. Below we assume that the continuous spectrum fills $[0, \infty)$ and use for illustration a particular model of the q-oscillator where one has only the continuous spectrum. Such a model is built upon the free non-relativistic particle with the following generators of the q-harmonic oscillator algebra [12]:

$$\begin{aligned} L &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} = a^+a - \frac{1}{1-q} \\ a &= T^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{\sqrt{1-q}}\right) \qquad a^+ = \left(-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{\sqrt{1-q}}\right)T \end{aligned} \tag{42}$$

where T is the unitary dilation operator $T\psi(x) = q^{1/4}\psi(q^{1/2}x)$. In this model the vacuum $|0\rangle$ and q-Fock states (9) do not exist, because the zero mode of the operator a is not normalizable on the whole line.

The eigenvalue problem $L_{\mathrm{int}}\psi = \lambda\psi$ for the Hamiltonian (25) in the realization (42) is given by the linear mixed differential-q-difference equation (the primes denote derivatives with respect to the arguments of the functions):

$$\begin{aligned} -\psi''(x) &+ j^* q^{-1/4}\psi'(q^{-1/2}x) - jq^{3/4}\psi'(q^{1/2}x) \\ &+ \frac{1}{\sqrt{1-q}}\left(jq^{1/4}\psi(q^{1/2}x) + j^* q^{-1/4}\psi(q^{-1/2}x)\right) \\ &+ Jq^{1/2}\left(q^{3/2}\psi''(qx) - \frac{q+q^{1/2}}{\sqrt{1-q}}\psi'(qx) + \frac{1}{1-q}\psi(qx)\right) \\ &+ Jq^{-1/2}\left(q^{-1/2}\psi''(q^{-1}x) + \frac{1+q^{-1/2}}{\sqrt{1-q}}\psi'(q^{-1}x) + \frac{1}{1-q}\psi(q^{-1}x)\right) \\ &= \lambda\psi(x). \end{aligned} \tag{43}$$

This expression looks quite cumbersome. However, some properties of the spectrum can be unravelled owing to the hidden symmetry. Similar types of equations were considered in [21] from a different point of view.

Following [12], we may substitute into (43) the Fourier transformation

$$\psi(x) = \int_{-\infty}^{\infty} \phi(p) \mathrm{e}^{\mathrm{i}px} \mathrm{d}p$$

which is well defined for functions from $L^2(R)$, and arrive at the equation:

$$\begin{aligned} & j^* q^{1/4} \left(\mathrm{i}pq^{1/2} + \frac{1}{\sqrt{1-q}} \right) \phi(q^{1/2}p) + jq^{-1/4} \left(-\mathrm{i}p + \frac{1}{\sqrt{1-q}} \right) \phi(q^{-1/2}p) \\ & \quad + (p^2 - \lambda)\phi(p) + Jq^{1/2} \left(-q^{3/2}p^2 + \frac{q+q^{1/2}}{\sqrt{1-q}} \mathrm{i}p + \frac{1}{1-q} \right) \phi(qp) \\ & \quad + Jq^{-1/2} \left(-q^{-1/2}p^2 - \frac{1+q^{-1/2}}{\sqrt{1-q}} \mathrm{i}p + \frac{1}{1-q} \right) \phi(q^{-1}p) = 0. \end{aligned} \tag{44}$$

This is a q-difference equation of the fourth order which is not easy to analyse for arbitrary values of parameters.

Let us look for eigenstates of the abstract Hamiltonian L_{int} in the form of the integral expansion over the continuous spectrum states of the free system $|\lambda\rangle$:

$$|\psi\rangle^{\mathrm{int}} = \int_0^{\infty} B(\lambda)|\lambda\rangle \, \mathrm{d}\lambda. \tag{45}$$

These states are normalizable when

$$\int_0^{\infty} \lambda |B(\lambda)|^2 \, \mathrm{d}\lambda < \infty. \tag{46}$$

Consider zero modes of the operators A_ℓ. For the free non-relativistic particle model of the q-oscillator they are determined by the generalized pantograph equation (cf [22, 23]):

$$\begin{aligned} & \alpha_\ell q^{-1/4} \left(\psi'(q^{-1/2}x) + \frac{1}{\sqrt{1-q}} \psi(q^{-1/2}x) \right) \\ & \quad + \beta_\ell q^{1/4} \left(-q^{1/2}\psi'(q^{1/2}x) + \frac{1}{\sqrt{1-q}} \psi(q^{1/2}x) \right) = -\gamma_\ell \psi(x). \end{aligned} \tag{47}$$

We set $\gamma_\ell = 0$ and assume that these zero modes can be represented in the form (45). Substituting (45) into (16) we come to the following equation for $B(\lambda)$

$$\alpha_\ell q \sqrt{\lambda q + \frac{1}{1-q}} B(q\lambda) + \frac{\beta_\ell}{q} \sqrt{\lambda + \frac{1}{1-q}} B(q^{-1}\lambda) = 0. \tag{48}$$

Solutions of (48) are defined up to the multiplication by an arbitrary function $g(\lambda)$ periodic in the logarithmic scale, $g(q^2\lambda) = g(\lambda)$. Fourier expanding this function in the variable $\ln\lambda$, we find that there is a countable infinity of states $|0\rangle_\ell^{\mathrm{int}}$:

$$|0; s\rangle_\ell^{\mathrm{int}} = C \int_0^{\infty} \lambda^{\rho_s} \sqrt{\frac{((q-1)q^2\lambda; q^2)_\infty}{((q-1)q\lambda; q^2)_\infty}} |\lambda\rangle \, \mathrm{d}\lambda \tag{49}$$

where the notation $(a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$ is used, and

$$\begin{aligned} & \rho_s = \frac{\pi\mathrm{i}(2s+1) + \ln(\beta_\ell/\alpha_\ell q^2)}{\ln q^2} \qquad s = 0, \pm 1, \pm 2, \ldots \\ & |C|^{-2} = \int_0^{\infty} \lambda^{\tau-1} \frac{((q-1)q^2\lambda; q^2)_\infty}{((q-1)q\lambda; q^2)_\infty} \, \mathrm{d}\lambda \qquad \tau = \frac{\ln|\beta_\ell/\alpha_\ell|}{\ln q}. \end{aligned} \tag{50}$$

Using the Ramanujan q-beta integral [19], we can find the explicit form of the normalization constant

$$|C|^{-2} = \frac{\Gamma(\tau)\Gamma(1-\tau)(q;q^2)_\infty(q^{2-2\tau};q^2)_\infty}{q^\tau(1-q)^\tau(q^2;q^2)_\infty(q^{1-2\tau};q^2)_\infty}$$

where $\Gamma(\tau)$ is the standard Γ-function. The integral (50) converges near $\lambda = 0$ if

$$|\beta_\ell/\alpha_\ell| < 1. \tag{51}$$

Integrability near the $\lambda = \infty$ point is guaranteed if

$$|\beta_\ell/\alpha_\ell| > \sqrt{q}. \tag{52}$$

We thus come to the following statement.

Proposition 4. The states $|0; s\rangle_\ell^{\text{int}}$ (49) belong to the Hilbert space in the domain

$$q^{\ell+1/2} < |\beta_0/\alpha_0| < q^\ell \qquad \text{or} \qquad -\ell - \tfrac{1}{2} < \sigma < -\ell.$$

There is thus only one normalizable eigenstate of the Hamiltonian H (for $\gamma = 0$) of infinite multiplicity $|\ell; s\rangle_0^{\text{int}}$ coming from the first factorization scheme. The second scheme does not yield bound states.

Using the connection between σ_1 and J we see that if

$$\frac{q^{\ell+1/2}}{1+q^{2\ell+2}} < |J| < \frac{q^\ell}{1+q^{2\ell+1}}$$

for some integer $\ell \geqslant 0$, then L_{int} has a bound state of infinite multiplicity.

Unusual types of spectra appear due to the exotic operator algebras and equations that are involved in the definition of our spectral problems. For example, the differential-difference equations associated with the free non-relativistic particle realization of the q-oscillator algebra admit an infinite number of linearly independent normalizable solutions. Such a situation is reached because one sacrifices the analyticity of the corresponding functions at the $x = 0$ point.

5. Comments on the continuous spectrum

The spectrum of the Hamiltonian (1) for $\gamma = 0$ and $0 < q < 1$ contains a finite number of discrete points for any nonzero α and β. If the 'free' q-oscillator admits q-Fock states then a finite number of superpositions of these states form discrete energy levels of the interacting system and the 'rest' of superpositions form a band of the continuous energy states [14]. This band exists independently on the presence of the continuous part in the spectrum of the 'free' q-oscillator. Let us consider briefly its structure in the present model. Suppose that there are non-degenerate continuous spectrum states $|E\rangle$

$$L_{\text{int}}|E\rangle = E|E\rangle \qquad \langle E_1|E_2\rangle = \delta(E_1 - E_2) \tag{53}$$

such that

$$|E\rangle = \sum_{n=0}^{\infty} B_n|n\rangle \tag{54}$$

where $|n\rangle$ are the q-Fock states (9).

Substituting (54) into (53) and taking the Hamiltonian in the form (25) with $j = 0$, one comes to the following recurrence relation

$$\left(\frac{q^n}{q-1} - E\right)B_n + J\left(\sqrt{[n-1]_q[n]_q}B_{n-2} + \sqrt{[n+1]_q[n+2]_q}B_{n+2}\right) = 0 \tag{55}$$

where $n = 0, 1, 2, \ldots$. The initial conditions $B_{-2} = B_{-1} = 0$ are imposed in (55) and the coefficients B_0 and B_1 are kept free. In the limit $n \to \infty$ equation (55) is reduced to

$$B_{n-2} - \frac{E(1-q)}{J} B_n + B_{n+2} = 0. \tag{56}$$

The general solution of this equation has the form $B_n = c_1\zeta_1^n + c_2\zeta_2^n + c_3\zeta_3^n + c_4\zeta_4^n$, where ζ_i stand for the roots of the characteristic equation

$$\zeta^4 - \frac{E(1-q)}{J}\zeta^2 + 1 = 0. \tag{57}$$

Solving (57) we obtain

$$\zeta^2 = \frac{E(1-q)}{2J} \pm \sqrt{\left(\frac{E(1-q)}{2J}\right)^2 - 1}. \tag{58}$$

The coefficients B_n are bounded when all roots (58) lie on the unit circle, $|\zeta_i| = 1$, i.e. when

$$|E| < \frac{2|J|}{1-q}. \tag{59}$$

If this is the case, the initial conditions can be satisfied by particular combinations of the linearly independent solutions. Thus, our heuristic considerations suggest that there is a band of continuous spectrum in the energy range (59).

In conclusion, let us mention that the interacting system has a continuous spectrum formed by the 'free' q-oscillator continuous spectrum states (in the realizations where the latter exist). There are, however, some difficulties with the precise characterization of this type of spectrum [14], and we do not consider it in the present work. In this and several other aspects our results are not complete. However, they give some basis for further improvements and more rigorous considerations in the future.

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